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General solutions to fundamental problems of $SU(2)_k$ WZW models

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Abstract. In this paper we give solutions to fundamental problems of the Wess-Zumino-Witten model for general cases. We find also the explicit connection between the crossing matrices of the $SU(2)_k$ WZW model and the crossing matrices of the SL(2, q) quantum group; the latter are the quantum Racah coefficients.

1. Introduction

In recent years the study of conformal field theory has developed a great deal [1-3]. As we know, the $SU(2)_k$ WZW model is connected with the minimal [1] model, the superconformal model [2] and the parafermionic model [4] through the GKO coset constructions [3]. For example, the constructions $SU(2)_k \otimes SU(2)_l/SU(2)_{k+l}$, when $l=1$, correspond to the minimal model, $l=2$ corresponds to the superconformal model, $l \geqslant 3$ corresponds to the parafermionic model. So study of the $SU(2)_k$WZW model is not only helpful to understanding the model itself, but also helpful for understanding the theories of the minimal model, superconformal model and parafermionic model etc. Recently, Alvarez-Gaumé *et al* found that the $SU(2)_k$WZW model is related to the SL(2, q) quantum group [5]. In this paper we will make this relation clearly and explicitly.

A few years ago, Knizhnik and Zamolodchikov discussed the SU(n) WZW model and provided an equation satisfied by the correlation functions (the so-called KZ equation) [6]. Tsuchiya and Kanie [7], using the approach of vertex operators, studied the $SU(2)_k$ WZW model. They identified a fundamental problem of the model, namely to determine the crossing (braid or fusion) matrices of the model. They gave a solution to the reduced KZ equation for the case of isospin $j_3=\frac{1}{2}$, and worked out the solution to the fundamental problem for that case. They did not give general solutions, mainly because of the difficulty of solving third-order or higher-order differential equations. From this paper we find that the crossing matrices do not depend on the detailed forms of the solutions to the reduced KZ equation, it depends on the exponents of the differential equations at singular points. Thus, we need not solve the KZ differential equations. In the present paper, solution of the fundamental problem for the case $j_3=1$ is worked out as an example of our method. The general solutions are given thereafter.

A brief review of the fundamental problem is given in the next section. The KCG constraint to the model is discussed in section 3. Solutions to fundamental problems for $j_3=1$ and general cases are given in section 4. In section 5 we provide an explicit connection between the $SU(2)_k$ WZW model and SL(2, q) quantum group, namely that

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the crossing matrices of the $SU(2)_k$ WZW model and those of the $SL(2, q)$ quantum group are connected by triangle matrix transformations which reduce to similar transformations when $j_1 = j_4$.

2. The fundamental problem of the model

For the sake of completeness, we had better give a brief review of the fundamental problem of the $SU(2)_k$ WZW model.

A triple $\binom{j}{j_2 j_1}$ of non-negative half integers j_2, j_1 and j is called a vertex which presents a process of coupling $j \otimes j_2 \to j_1$. In [7] it was proven that there exists a non-zero vertex Φ of type $\binom{j}{j_2 j_1}$ on $\mathscr{H}$ if and only if the vertex $\binom{j}{j_2 j_1}$ is a KCG vertex, i.e. it satisfies the K-constrained Clebsh-Gordan conditions:

$$|j_2 - j| \leq j_1 \leq j_1 + j \qquad j_1 + j_2 + j \leq k \qquad j_1 + j_2 + j \in \mathbb{Z} \tag{2.1}$$

where $\mathscr{H}$ is the highest-weight space of an affine Lie algebra of type $A_1^{(1)}$.

The KZ equations for the $SU(2)_k$ WZW model have the forms

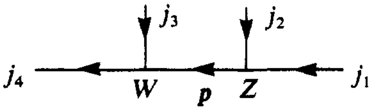
$$\left((k+2)\frac{\partial}{\partial z_i} - \sum_{j \neq i} \frac{\Omega_{ij}}{z_i - z_j}\right)\langle \Phi_N(z_N) \ldots \Phi_1(z_1)\rangle = 0 \tag{2.2}$$

where

$$\Omega_{ij} = \sum_a t_i^a t_j^a, \tag{2.3}$$

and t_i^a are the representations of the SU(2) generators of the field Φ_i.

For a quadruple $(j_4 j_3 j_2 j_1)$ of half integers, we have two vertices $\Phi_1(w)$, $\Phi_2(z)$ which correspond to $\binom{j_3}{j_4 p}$ and $\binom{j_2}{p j_1}$, respectively (as shown in the figure below):



The correlation function $\psi(w, z) = \langle \Phi_1(w)\Phi_2(z)\rangle$ satisfies the equation

$$\begin{aligned} &\left((k+2)\frac{\partial}{\partial w} - \frac{\Omega_{13}}{w} - \frac{\Omega_{23}}{w-z}\right)\psi(w, z) = 0 \\ &\left((k+2)\frac{\partial}{\partial z} - \frac{\Omega_{12}}{z} - \frac{\Omega_{23}}{z-w}\right)\psi(w, z) = 0 \end{aligned} \tag{2.4}$$

which is just the KZ equation for the case $z_1 = 0$, $z_2 = z$, $z_3 = w$ and $z_4 = \infty$. From (2.4) we also find

$$\left(w\frac{\partial}{\partial w} + z\frac{\partial}{\partial z} + \Delta_4\right)\psi(w, z) = 0 \tag{2.5}$$

where

$$-(k+2)\Delta_4 = \Omega_{12} + \Omega_{13} + \Omega_{23}.$$

Introduce a variable $\zeta = z/w$. From (2.5) we find that the function $z^{\Delta_4}\psi(w, zw)$ is independent of w. So, we abbreviate $z^{\Delta_4}\psi(w, zw)$ to $\psi(\zeta)$. Then, from (2.4) we obtain the reduced KZ equation

$$\left((k+2)\frac{d}{d\zeta} - \frac{\Omega_{12}+(k+2)\Delta_4}{\zeta} - \frac{\Omega_{23}}{\zeta-1}\right)\psi(\zeta) = 0. \tag{2.6}$$

The change of coordinate $\zeta \to \eta = 1/\zeta$ transforms equation (2.6) into

$$\left((k+2)\frac{d}{d\eta} - \frac{\Omega_{13}}{\eta} - \frac{\Omega_{23}}{\eta-1}\right)\psi(1/\eta) = 0. \tag{2.7}$$

For any quadruple $(j_4 j_3 j_2 j_1)$, the function $\psi(\zeta)$ is V_0 valued, the space V_0 is the space of all SL(2, c) invariant elements in $V_{j_4} \otimes V_{j_3} \otimes V_{j_2} \otimes V_{j_1}$, and V_j is the dual SL(2, c)-modules of the irreducible left SL(2, c) module of isospin j. There are three orthogonal bases $\{U^{(0)}_{j_{12}}\}\{U^{(1)}_{j_{23}}\}\{U^{(\infty)}_{j_{13}}\}$ of V_0, defined by

$$U^{(0)}_{j_{12}} = \frac{1}{\sqrt{2j_4+1}} \sum_{\substack{m_1+m_2=m_{12}\\ m_{12}+m_3=m_4}} (-1)^{j_4-m_4} C^{j_{12}m_{12}}_{m_1m_2} C^{j_4m_4}_{m_{12}m_3} \varphi_{j_4}(-m_4)\otimes\varphi_{j_3}(m_3)\otimes\varphi_{j_2}(m_2)\otimes\varphi_{j_1}(m_1)$$

$$U^{(1)}_{j_{23}} = \frac{1}{\sqrt{2j_4+1}} \sum_{\substack{m_2+m_3=m_{23}\\ m_{23}+m_1=m_4}} (-1)^{j_4-m_4} C^{j_4m_4}_{m_1m_{23}} C^{j_{23}m_{23}}_{m_2m_3} \varphi_{j_4}(-m_4)\otimes\varphi_{j_3}(m_3)\otimes\varphi_{j_2}(m_2)\otimes\varphi_{j_1}(m_1)$$

$$\tag{2.8}$$

$$U^{(\infty)}_{j_{13}} = \frac{1}{\sqrt{2j_4+1}} \sum_{\substack{m_1+m_3=m_{13}\\ m_2+m_{13}=m_4}} (-1)^{j_4-m_4} C^{j_4m_4}_{m_2m_{13}} C^{j_{13}m_{13}}_{m_1m_3} \varphi_{j_4}(-m_4)\otimes\varphi_{j_3}(m_3)\otimes\varphi_{j_2}(m_2)\otimes\varphi_{j_1}(m_1)$$

where $C^{j_3m_3}_{m_1m_2}$ are CG coefficients, and $\varphi_j^{(m)}$ are elements of the base of V_j. The three bases of V_0 defined in (2.8) diagonalise the operators Ω_{12}, Ω_{23} and Ω_{13} respectively, i.e.

$$\begin{aligned}
\Omega_{12}U^{(0)}_{j_{12}} &= (k+2)(\Delta_{j_{12}} - \Delta_{j_2} - \Delta_{j_1})\,U^{(0)}_{j_{12}}\\
\Omega_{23}U^{(1)}_{j_{23}} &= (k+2)(\Delta_{j_{23}} - \Delta_{j_2} - \Delta_{j_3})\,U^{(1)}_{j_{23}}\\
\Omega_{13}U^{(\infty)}_{j_{13}} &= (k+2)(\Delta_{j_{13}} - \Delta_{j_1} - \Delta_{j_3})\,U^{(\infty)}_{j_{13}}
\end{aligned} \tag{2.9}$$

where $\Delta_j = j(j+1)/(k+2)$.

Write a solution to equation (2.6) as

$$\psi(\zeta) = U^{(i)}\psi^{(i)}(\zeta) = U^{(\infty)}\psi^{(\infty)}(1/\zeta) \qquad i = 0, 1 \tag{2.10}$$

where $U^{(0)}\psi^{(0)}(\zeta)$, $U^{(1)}\psi^{(1)}(\zeta)$ and $U^{(\infty)}\psi^{(\infty)}(\zeta)$ are the solutions to (2.6) under the bases $U^{(0)}$, $U^{(1)}$ and $U^{(\infty)}$, respectively. And the three solutions analyse at $\zeta = 0$, $\zeta = 1$ and $\zeta = \infty$, respectively. There are two transformation relations among the three solutions:

$$U^{(0)}\psi^{(0)} = U^{(\infty)}\psi^{(\infty)}K(j_4j_3j_2j_1) \qquad U^{(0)}\psi^{(0)} = U^{(1)}\psi^{(1)}F(j_4j_3j_2j_1). \tag{2.11}$$

We denote the transformation matrices among the three bases as S, i.e.

$$U^{(0)} = U^{(\infty)}S^{(0,\infty)} \qquad U^{(0)} = U^{(1)}S^{(0,1)}. \tag{2.12}$$

Inserting (2.12) into (2.11), we obtain

$$S^{(0,\infty)}\psi^{(0)}(\zeta) = \psi^{(\infty)}(\zeta)K(j_4j_3j_2j_1) \qquad S^{(0,1)}\psi^{(0)}(\zeta) = \psi^{(1)}(\zeta)F(j_4j_3j_2j_1). \tag{2.13}$$

The important thing is that the transformation matrices (F or K) among the solutions under the different bases are independent of ζ. The so-called fundamental problem is to determine the transformation matrices K and F. As we know, however, the matrices K and F are just the crossing matrices (braid and fusion matrices) of the model.

3. KCG constraint condition of the model

For a quadruple $(j_4 j_3 j_2 j_1)$ with $j_3 = \min(j_4 j_3 j_2 j_1)$, we introduce a set $I_k(j_4 j_3 j_2 j_1)$ defined by

$$I_k(j_4 j_3 j_2 j_1) = \{p \in \tfrac{1}{2}\mathbb{Z}_{\geq 0}, 0 \leq 2p \leq k, \left(\begin{smallmatrix} & j_3 & \\ j_4 & & p\end{smallmatrix}\right) \in (\text{KCG}), \left(\begin{smallmatrix} & j_2 & \\ p & & j_1\end{smallmatrix}\right) \in (\text{KCG})\} \tag{3.1}$$

where (KCG) denotes the set of vertex operators which satisfy the KCG constraint conditions. The definition of the set I_k means that the values of intermediate edges p should be constrained by the following four constraint equations:

$$\left.\begin{array}{l} j_4 - j_3 \leq p \leq j_4 + j_3 \\ |j_1 - j_2| \leq p \leq j_1 + j_2 \end{array}\right\} \quad \text{CG constraint}$$
$$\left.\begin{array}{l} j_4 + j_3 + p \leq k \\ j_1 + j_2 + p \leq k \end{array}\right\} \quad k \text{ constraint.} \tag{3.2}$$

This is just the KCG constraint for the conformal block $\psi(\zeta)$ in (2.6).

Given $j_4 - j_3 \geq |j_1 - j_2|$, $j_4 + j_3 \leq j_2 + j_1$ and k sufficiently large that $k \geq j_1 + j_2 + j_3 + j_4$, then p takes values in the set

$$I_k = \{j_4 + j_3, j_4 + j_3 - 1, \ldots, j_4 - j_3\}. \tag{3.3}$$

Under this case, the reduced KZ equation will become $2j_3 + 1$ linear differential equations and the matrices K and F become $2j_3 + 1 \times 2j_3 + 1$ matrices. But for some special cases the CG-admissible values of p will reduce and hence the dimension of V_0 (number of p) will be reduced as well. We denote the case under which the admissible p have m values as (DM).

When $j_1 = j_2 + j_4 + j_3 - m + 1$, or $j_2 = j_1 + j_4 + j_3 - m + 1$, or $j_4 = j_1 + j_2 + j_3 - m + 1$, under the CG constraint, the $2j_3 + 1$ values of p for every case above will be reduced to m admissible values, the space V_0 becomes m dimensional. We denote these three cases as $(DM)^1$, $(DM)^2$ and $(DM)^3$, respectively.

For $(DM)^{1,2}$

$$I_k = \{j_4 + j_3, j_4 + j_3 - 1, \ldots, j_4 + j_3 - m + 1\} \tag{3.4}$$

and for $(DM)^3$

$$I_k = \{j_4 - j_3 + m - 1, j_4 - j_3 + m - 2, \ldots, j_4 - j_3\} \tag{3.5}$$

where m takes values in the region $1 \leq m \leq 2j_3$.

Of course, the above consideration is only of the CG constraint. If k takes certain values in the region $j_1 + j_2 + j_4 - j_3 < k < j_1 + j_2 + j_3 + j_4$, the k constraint should be taken into consideration. Under the k constraint the cases $(DM)^{1,2,3}$ will degenerate to lower-dimension cases (say n dimensional), we denote this case as $(DM)_n^{1,2,3}$ $(n < m)$.

For example, when $k=j_1+j_2+j_4+j_3-n$, under the k constraint, the cases $(DM)^{1,2,3}$ degnerate to $(DM)^{1,2,3}_{m-n}$ $(0\leqslant n\leqslant m)$. The intermediate edges p take values in

$$I_k=\begin{cases}\{j_4+j_3-m+1, j_4+j_3-m+2,\dots,j_4+j_3-n\} & \text{for } (DM)^{1,2}_{m-n} \quad (3.6)\\ \{j_4-j_3+m-n, j_4-j_3+m-n+1,\dots,j_4-j_3\} & \text{for } (DM)^{3}_{m-n}. \quad (3.7)\end{cases}$$

$I_k=\varnothing$ when $m=n$.

4. Solution of the fundamental problems

The solution of the fundamental problem for the case $j_3=\frac{1}{2}$ was given in [7], so we need not repeat it here. In this section we consider the case $j_3=1$ first, then we give the general solutions.

For the quadruple $(j_4 1 j_2 j_1)$ with $j_4, j_2, j_1\geqslant 1$ we have

$$I_k(j_4 1 j_2 j_1)=\{p\in\tfrac{1}{2}\mathbb{Z}_{\geqslant 0}, \left(\begin{smallmatrix} & 1 & \\ j_4 & & p\end{smallmatrix}\right)\in(\text{KCG}), \left(\begin{smallmatrix} & j_2 & \\ p & & j_1\end{smallmatrix}\right)\in(\text{KCG})\}. \tag{4.1}$$

We know that the number of p in I_k is equal to dim $V_0\leqslant 2j_3+1=3$, and dim $V_0=3$ if and only if (case $(D3)$)

$$j_1-j_2\leqslant j_4-1 \qquad j_4+1\leqslant j_1+j_2 \qquad 1+j_1+j_2+j_4\in\mathbb{Z}. \tag{4.2}$$

From the discussion in section 3, we know that the case $(D3)$ can degenerate to $(D2)^{1,2,3}$ and $(D1)^{1,2,3}$ under the special choice of j_i $(i=1,2,4)$.

First we investigate some basic problems in case $(D3)$. We can see that any of the three bases of V_0 defined in section 2 has three elements in case $(D3)$. j_{12} can be j_4+1, j_4, j_4-1 and we denote these three elements in $\{U^{(0)}_{j_{12}}\}$ as $U^{(0)}_i$ $(i=1,2,3)$. We can also write $\{U^{(1)}_{j_{23}}\}$ and $\{U^{(\infty)}_{j_{13}}\}$ as $U^{(1)}_i$ and $U^{(\infty)}_i$, where $i=1, 2, 3$ correspond to $j_{23}=j_2+1$, j_2, j_2-1 and $j_{13}=j_1+j_1, j_1, j_1-1$, respectively.

We define

$$\begin{aligned}&\Omega_{12}u^{(0)}_i=(k+2)(\gamma^{(0)}_i-\Delta_4)u^{(0)}_i \qquad \Omega_{23}u^{(1)}_i=(k+2)\gamma^{(1)}_i u^{(1)}_i\\ &\Omega_{13}u^{(\infty)}_i=(k+2)\gamma^{(\infty)}_i u^{(\infty)}_i\end{aligned} \tag{4.3}$$

where

$$\Delta_4=\Delta_{j_1}+\Delta_{j_2}+\Delta_{j_3}-\Delta_{j_4}.$$

So we get

$$\begin{aligned}&\gamma^{(0)}_1=\frac{1}{k+2}(2j_4+4) \qquad \gamma^{(0)}_2=\frac{2}{k+2} \qquad \gamma^{(0)}_3=-\frac{2j_4-2}{k+2}\\ &\gamma^{(1)}_1=\frac{1}{k+2}2j_2 \qquad \gamma^{(1)}_2=-\frac{2}{k+2} \qquad \gamma^{(1)}_3=-\frac{2j_2+2}{k+2}\\ &\gamma^{(\infty)}_1=\frac{1}{k+2}2j_1 \qquad \gamma^{(\infty)}_2=-\frac{2}{k+2} \qquad \gamma^{(\infty)}_3=-\frac{2j_1+2}{k+2}.\end{aligned} \tag{4.4}$$

Introducing the differences

$$\eta^{(i)}_1=\gamma^{(i)}_1-\gamma^{(i)}_2 \qquad \eta^{(i)}_2=\tfrac{1}{2}(\gamma^{(i)}_1-\gamma^{(i)}_3) \qquad \eta^{(i)}_3=\gamma^{(i)}_2-\gamma^{(i)}_3 \qquad i=0,1,\infty \tag{4.5}$$

we have

$$\eta_1^{(0)}=\frac{2j_4+2}{k+2}\qquad \eta_1^{(1)}=\frac{2j_2+2}{k+2}\qquad \eta_1^{(\infty)}=\frac{2j_1+2}{k+2}$$
$$\eta_2^{(0)}=\frac{2j_4+1}{k+2}\qquad \eta_2^{(1)}=\frac{2j_2+1}{k+2}\qquad \eta_2^{(\infty)}=\frac{2j_1+1}{k+2} \tag{4.6}$$
$$\eta_3^{(0)}=\frac{2j_4}{k+2}\qquad \eta_3^{(1)}=\frac{2j_2}{k+2}\qquad \eta_3^{(\infty)}=\frac{2j_1}{k+2}.$$

The transformation matrices $S^{(lk)}$ between $\{U_i^{(l)}\}$ and $\{U_i^{(l)}\}$ are given by

$$(u_1^{(k)},u_2^{(k)},u_3^{(k)})=(u_1^{(l)},u_2^{(l)},u_3^{(l)})S^{(lk)}. \tag{4.7}$$

Using $6j$ symbols from [9] we get

$$S^{(\infty 0)}={}^{\mathrm{t}}S^{(0\infty)}=\begin{pmatrix} A_{11} & -A_{12} & A_{13}\\ -A_{21} & A_{22} & A_{23}\\ A_{31} & A_{32} & A_{33}\end{pmatrix}$$
$$S^{(10)}={}^{\mathrm{t}}S^{(01)}=\begin{pmatrix} A'_{11} & A'_{12} & A'_{13}\\ A'_{21} & A'_{22} & -A'_{23}\\ A'_{31} & -A'_{32} & A'_{33}\end{pmatrix} \tag{4.8}$$
$$S^{(\infty 1)}={}^{\mathrm{t}}S^{(1\infty)}=\begin{pmatrix} A''_{11} & -A''_{12} & A''_{13}\\ A''_{21} & -A''_{22} & -A''_{23}\\ A''_{31} & A''_{32} & A''_{33}\end{pmatrix}$$

$$A_{11}=\left(\frac{\varepsilon_1\varepsilon'_1\varepsilon_4\varepsilon'_4}{\eta_1^{(0)}\eta_2^{(0)}\eta_1^{(\infty)}\eta_2^{(\infty)}}\right)^{1/2}\qquad A_{12}=\left(\frac{2\varepsilon_0\varepsilon'_0\varepsilon_2\varepsilon_4}{\eta_1^{(0)}\eta_3^{(0)}\eta_2^{(\infty)}\eta_1^{(\infty)}}\right)^{1/2}$$
$$A_{13}=\left(\frac{\varepsilon_0\varepsilon'_0\varepsilon_2\varepsilon'_2}{\eta_2^{(0)}\eta_3^{(0)}\eta_1^{(\infty)}\eta_2^{(\infty)}}\right)^{1/2}\qquad A_{21}=\left(\frac{2\varepsilon_0\varepsilon_1\varepsilon_2\varepsilon'_4}{\eta_1^{(0)}\eta_2^{(0)}\eta_1^{(\infty)}\eta_3^{(\infty)}}\right)^{1/2}$$
$$A_{22}=\frac{\varepsilon_0\varepsilon'_2-\varepsilon'_1\varepsilon'_4}{\sqrt{\eta_1^{(0)}\eta_3^{(0)}\eta_1^{(\infty)}\eta_3^{(\infty)}}}\qquad A_{23}=\left(\frac{2\varepsilon'_0\varepsilon'_1\varepsilon'_2\varepsilon_4}{\eta_2^{(0)}\eta_3^{(0)}\eta_1^{(\infty)}(\eta_3^{(\infty)}}\right)^{1/2}$$
$$A_{31}=\left(\frac{\varepsilon_0\varepsilon'_0\varepsilon_2\varepsilon'_2}{\eta_1^{(0)}\eta_2^{(0)}\eta_2^{(\infty)}\eta_3^{(0)}}\right)^{1/2}\qquad A_{32}=\left(\frac{2\varepsilon'_0\varepsilon_1\varepsilon'_2\varepsilon'_4}{\eta_1^{(0)}\eta_3^{(0)}\eta_2^{(\infty)}\eta_3^{(\infty)}}\right)^{1/2}$$
$$A_{33}=\left(\frac{\varepsilon_1\varepsilon'_1\varepsilon_4\varepsilon'_4}{\eta_2^{(0)}\eta_3^{(0)}\eta_2^{(\infty)}\eta_3^{(\infty)}}\right)^{1/2}$$

and

$$A'_{ij}=A_{ij}\quad (j_2\rightleftharpoons j_1)$$
$$A''_{ij}=A_{ij}\quad (j_2\rightleftharpoons j_4).$$

The ε_i are given by

$$\varepsilon_0=\frac{1}{k+2}(j_1+j_2+j_4+2)\qquad \varepsilon_i=\frac{1}{k+2}(j_1+j_2+1+j_4-2j_i)\quad i=1,2,4$$
$$\varepsilon'_0=\frac{1}{k+2}(j_1+j_2+j_4+1)\qquad \varepsilon'_i=\frac{1}{k+2}(j_1+j_2+j_4-2j_i)\quad i=1,2,4. \tag{4.9}$$

Write a solution $\psi(\zeta)$ as

$$\psi(\zeta)=(u_1^{(i)},\, u_2^{(i)},\, u_3^{(i)})\psi^{(i)}(\zeta)=(u_1^{(i)},\, u_2^{(i)}u_3^{(i)})\begin{pmatrix}\varphi_{11}^{(i)} & \varphi_{12}^{(i)} & \varphi_{13}^{(i)}\\ \varphi_{22}^{(i)} & \varphi_{23}^{(i)} & \varphi_{24}^{(i)}\\ \varphi_{31}^{(i)} & \varphi_{32}^{(i)} & \varphi_{33}^{(i)}\end{pmatrix}. \tag{4.10}$$

Equation (2.6) becomes

$$\frac{\mathrm{d}}{\mathrm{d}\zeta}(\varphi_{ij}^{(0)})=\begin{bmatrix}\frac{\gamma_1^{(0)}}{\zeta}+\frac{\gamma_1^{(1)}-a_1^{(0)}}{\zeta-1} & \frac{b_1^{(0)}}{\zeta-1} & 0\\ \frac{b_1^{(0)}}{\zeta-1} & \frac{\gamma_2^{(0)}}{\zeta}+\frac{\gamma_2^{(1)}+a_2^{(0)}}{\zeta-1} & \frac{b_2^{(0)}}{\zeta-1}\\ 0 & \frac{b_2^{(0)}}{\zeta-1} & \frac{\gamma_3^{(0)}}{\zeta}+\frac{\gamma_3^{(1)}+a_3^{(0)}}{\zeta-1}\end{bmatrix}(\varphi_{ij}^{(0)})$$

$$\frac{\mathrm{d}}{\mathrm{d}\zeta}(\varphi_{ij}^{(1)})=\begin{bmatrix}\frac{\gamma_1^{(0)}-a_1^{(1)}}{\zeta}+\frac{\gamma_1^{(1)}}{\zeta-1} & \frac{b_1^{(1)}}{\zeta} & 0\\ \frac{b_1^{(1)}}{\zeta} & \frac{\gamma_2^{(0)}+a_2^{(1)}}{\zeta}+\frac{\gamma_2^{(1)}}{\zeta-1} & \frac{b_2^{(1)}}{\zeta}\\ 0 & \frac{b_2^{(1)}}{\zeta} & \frac{\gamma_3^{(0)}+a_3^{(1)}}{\zeta}+\frac{\gamma_3^{(1)}}{\zeta-1}\end{bmatrix}(\varphi_{ij}^{(1)}) \tag{4.11}$$

and equation (2.7) becomes

$$\frac{\mathrm{d}}{\mathrm{d}\zeta}(\varphi_{ij}^{(\infty)})=\begin{bmatrix}\frac{\gamma_1^{(\infty)}}{\eta}+\frac{\gamma_1^{(1)}-a_1^{(\infty)}}{\eta-1} & \frac{b_1^{(\infty)}}{\eta-1} & 0\\ \frac{b_1^{(\infty)}}{\eta-1} & \frac{\gamma_2^{(\infty)}}{\eta}+\frac{\gamma_2^{(1)}+a_2^{(\infty)}}{\eta-1} & \frac{b_2^{(\infty)}}{\eta-1}\\ 0 & \frac{b_2^{(\infty)}}{\eta-1} & \frac{\gamma_3^{(\infty)}}{\eta}+\frac{\gamma_3^{(1)}+a_3^{(\infty)}}{\eta-1}\end{bmatrix}(\varphi_{ij}^{(\infty)}) \tag{4.12}$$

where

$$\begin{aligned}
&a_1^{(0)}=\frac{2\varepsilon_0\varepsilon_1}{\eta_1^{(0)}} \qquad a_2^{(0)}=\frac{2(\varepsilon_0\varepsilon_2'-\varepsilon_1'\varepsilon_4')}{(k+2)\eta_1^{(0)}\eta_3^{(0)}} \qquad a_3^{(0)}=\frac{2\varepsilon_0'\varepsilon_1'}{\eta_3^{(0)}}\\
&a_1^{(1)}=\frac{2\varepsilon_0\varepsilon_1}{\eta_1^{(1)}} \qquad a_2^{(1)}=\frac{2(\varepsilon_0\varepsilon_4'-\varepsilon_1'\varepsilon_2')}{(k+2)\eta_1^{(1)}\eta_3^{(1)}} \qquad a_3^{(1)}=\frac{2\varepsilon_0'\varepsilon_1'}{\eta_3^{(1)}}\\
&a_1^{(\infty)}=\frac{2\varepsilon_0\varepsilon_4}{\eta_1^{(\infty)}} \qquad a_2^{(\infty)}=\frac{2(\varepsilon_0\varepsilon_2'-\varepsilon_1'\varepsilon_4')}{(k+2)\eta_1^{(\infty)}\eta_3^{(\infty)}} \qquad a_3^{(\infty)}=\frac{2\varepsilon_0'\varepsilon_4'}{\eta_3^{(\infty)}}\\
&b_1^{(0)}=\frac{2\eta_3^{(0)}}{\eta_1^{(0)}}\left(\frac{\varepsilon_0\varepsilon_1\varepsilon_\nu'\varepsilon_4}{2\eta_3^{(0)}\eta_2^{(0)}}\right)^{1/2} \qquad b_2^{(0)}=\frac{2\eta_1^{(0)}}{\eta_3^{(0)}}\left(\frac{\varepsilon_0'\varepsilon_1'\varepsilon_2'\varepsilon_4}{2\eta_1^{(0)}\eta_2^{(0)}}\right)^{1/2}\\
&b_1^{(1)}=\frac{2\eta_3^{(1)}}{\eta_1^{(1)}}\left(\frac{\varepsilon_0\varepsilon_1\varepsilon_2'\varepsilon_4}{2\eta_3^{(1)}\eta_2^{(1)}}\right)^{1/2} \qquad b_2^{(1)}=\frac{2\eta_1^{(1)}}{\eta_3^{(1)}}\left(\frac{\varepsilon_0'\varepsilon_1'\varepsilon_2\varepsilon_4'}{2\eta_1^{(1)}\eta_2^{(1)}}\right)^{1/2}\\
&b_1^{(\infty)}=\frac{2\eta_3^{(\infty)}}{\eta_1^{(\infty)}}\left(\frac{\varepsilon_0\varepsilon_1'\varepsilon_2\varepsilon_4}{2\eta_3^{(\infty)}\eta_2^{(\infty)}}\right)^{1/2} \qquad b_2^{(\infty)}=\frac{2\eta_1^{(\infty)}}{\eta_3^{(\infty)}}\left(\frac{\varepsilon_0'\varepsilon_1\varepsilon_2'\varepsilon_4'}{2\eta_1^{(\infty)}\eta_2^{(\infty)}}\right)^{1/2}.
\end{aligned} \tag{4.13}$$

From (4.11), we have

$$\begin{aligned}
\frac{d}{d\zeta}\varphi_{11}^{(0)} &= \left(\frac{\gamma_1^{(0)}}{\zeta}+\frac{\gamma_1^{(1)}-a_1^{(0)}}{\zeta-1}\right)\varphi_{11}^{(0)}+\frac{b_1^{(0)}}{\zeta-1}\varphi_{21}^{(0)} \\
\frac{d}{d\zeta}\varphi_{21}^{(0)} &= \frac{b_1^{(0)}}{\zeta-1}\varphi_{11}^{(0)}+\left(\frac{\gamma_2^{(0)}}{\zeta}+\frac{\gamma_2^{(1)}+a_2^{(0)}}{\zeta-1}\right)\varphi_{21}^{(0)}+\frac{b_2^{(0)}}{\zeta-1}\varphi_{31}^{(0)} \\
\frac{d}{d\zeta}\varphi_{31}^{(0)} &= \frac{b_2^{(0)}}{\zeta-1}\varphi_{21}^{(0)}+\left(\frac{\gamma_3^{(0)}}{\zeta}+\frac{\gamma_3^{(1)}+a_3^{(0)}}{\zeta-1}\right)\varphi_{31}^{(0)}
\end{aligned} \tag{4.14}$$

and $(\varphi_{12}^{(0)}\varphi_{22}^{(0)}\varphi_{32}^{(0)})$ and $(\varphi_{13}^{(0)}\varphi_{23}^{(0)}\varphi_{33}^{(0)})$ satisfy the same equations as (4.14). It is difficult to solve equations such as (4.14), but we can get the exponents of equations for $\varphi_{11}^{(0)}$, $\varphi_{21}^{(0)}$, $\varphi_{31}^{(0)}$ at the singular points $\zeta=0,\ 1,\ \infty$. We denote the solution by the exponents of the equation at the singular points as

$$\begin{aligned}
\varphi_{11}^{(0)},\varphi_{12}^{(0)},\varphi_{13}^{(0)} &\in p\begin{Bmatrix} 0 & 1 & \infty \\ \gamma_1^{(0)} & \gamma_1^{(1)} & \gamma_1^{(\infty)} \\ 1+\gamma_2^{(0)} & \gamma_2^{(1)} & \gamma_2^{(\infty)} \\ 2+\gamma_3^{(0)} & \gamma_3^{(1)} & \gamma_3^{(\infty)} \end{Bmatrix} \\
\varphi_{21}^{(0)},\varphi_{22}^{(0)},\varphi_{23}^{(0)} &\in p\begin{Bmatrix} 0 & 1 & \infty \\ \gamma_2^{(0)} & \gamma_2^{(1)} & \gamma_2^{(\infty)} \\ 1+\gamma_3^{(0)} & \gamma_3^{(1)} & \gamma_3^{(\infty)} \\ 1+\gamma_1^{(0)} & \gamma_1^{(1)} & \gamma_1^{(\infty)} \end{Bmatrix} \\
\varphi_{31}^{(0)},\varphi_{32}^{(0)},\varphi_{33}^{(0)} &\in p\begin{Bmatrix} 0 & 1 & \infty \\ \gamma_3^{(0)} & \gamma_3^{(1)} & \gamma_3^{(\infty)} \\ 1+\gamma_2^{(0)} & \gamma_2^{(1)} & \gamma_2^{(\infty)} \\ 2+\gamma_1^{(0)} & \gamma_1^{(1)} & \gamma_1^{(\infty)} \end{Bmatrix}.
\end{aligned} \tag{4.15}$$

Similar expressions can be obtained for $\varphi_{ij}^{(1)}$ and $\varphi_{ij}^{(\infty)}$.

In order to find the explicit form of the K (or F) matrix, we first need to solve the solutions of the cases $(D2)^{1,2,3}$ and $(D1)^{1,2,3}$ with the help of the knowledge of hypergeometric functions, and then we come back again to investigate the case $(D3)$. First we consider the case $(D2)^1$. Under this case, $j_1=j_2+j_4$, $I_k=\{j_4+I, j_4\}$ and ε_i, ε_i' become

$$\begin{aligned}
&\varepsilon_1'=0 \qquad \varepsilon_1=\frac{1}{k+2} \qquad \varepsilon_2=\frac{2j_4+1}{k+2} \qquad \varepsilon_2'=\frac{2j_4}{k+2} \\
&\varepsilon_4=\frac{2j_2+1}{k+2} \qquad \varepsilon_4'=\frac{2j_2}{k+2} \qquad \varepsilon_0=\frac{2j_1+2}{k+2} \qquad \varepsilon_0'=\frac{2j_1+1}{k+2}.
\end{aligned} \tag{4.16}$$

The transformation matrices (4.8) become

$$\begin{aligned}
&S^{(\infty 0)}=\begin{pmatrix} 0 & 0 & 1 \\ -A_{21} & A_{22} & 0 \\ A_{31} & A_{32} & 0 \end{pmatrix} \qquad S^{(10)}=\begin{pmatrix} A_{11}' & A_{12}' & 0 \\ A_{21}' & A_{22}' & 0 \\ 0 & 0 & 1 \end{pmatrix} \\
&S^{(\infty 1)}=\begin{pmatrix} 0 & -A_{12}'' & A_{13}'' \\ 0 & -A_{22}' & -A_{23}'' \\ 1 & 0 & 0 \end{pmatrix}.
\end{aligned} \tag{4.17}$$

So in this case, dim $V_0=2$, and it is convenient to choose the bases of V_0 as $\{u_1^{(0)}, u_2^{(0)}\}$, $\{u_1^{(1)}, u_2^{(1)}\}$, $\{u_1^{(\infty)}, u_2^{(\infty)}\}$, which correspond to $j_{12}=j_4+1$, j_4, $j_{23}=j_2+1$, j_2, and $j_{13}=j_1$, j_1-1, respectively. The transformation matrix $S^{(\infty 0)}$ for these new bases becomes

$$S^{(\infty,0)}=\begin{pmatrix} -A_{21} & A_{22} \\ A_{31} & A_{32} \end{pmatrix}_{j_1=j_4+j_2}$$

where

$$\begin{aligned} A_{21}|_{j_1=j_2+j_4} &= A_{32}|_{j_1=j_2+j_4}=\left(\frac{j_2}{j_1(j_4+1)}\right)^{1/2} \\ A_{22}|_{j_1=j_2+j_4} &= A_{31}|_{j_1=j_2+j_4}=\left(\frac{j_4(j_1+1)}{j_1(j_4+1)}\right)^{1/2}. \end{aligned} \tag{4.18}$$

With the help of the hypergeometric functions (see [7]) and after some calculations we can obtain the elements of the braid matrix K defined in (2.11). For the case $(D2)^1$, the matrix elements are

$$\begin{aligned} K_{11} &= -q^{-(j_4+1)}\left(\frac{2j_1(2j_4+2)}{2\times 2j_2}\right)^{1/2}\frac{\Gamma((2j_4+2)/(K+2))\Gamma(-2j_1/(K+2))}{\Gamma(2/(K+2))\Gamma(-2j_2/(K+2))} \\ &= -q^{-(j_4+1)}(A_{21}^{-1})\frac{\Gamma(\eta_1^{(0)})\Gamma(-\eta_3^{(0)})}{\Gamma(2(K+2))\Gamma(-\varepsilon_4')} \\ K_{21} &= q^{(j_2-1)}(A_{22}^{-1})\frac{\Gamma(\eta_1^{(0)})\Gamma(\eta_3^{(0)})}{\Gamma(\varepsilon_0)\Gamma(\varepsilon_2')} \\ K_{12} &= q^{0}(A_{31}^{-1})\frac{\Gamma(-\eta_1^{(0)})\Gamma(-\eta_3^{(0)})}{\Gamma(-\varepsilon_0)\Gamma(-\varepsilon_2')} \\ K_{22} &= q^{j_1}(A_{32}^{-1})\frac{\Gamma(-\eta_1^{(0)})\Gamma(\eta_3^{(\infty)})}{\Gamma(-2/(K+2))\Gamma(\varepsilon_4')} \end{aligned} \tag{4.19}$$

where $q=\exp[2\pi i/(k+2)]$.

After a similar discussion we can also obtain the braid matrix K for cases $(D2)^2$ and $(D2)^3$.

For $(D2)^2$, $j_2=j_1+j_4$, and the elements of the braid matrix for this case are

$$\begin{aligned} K_{11} &= q^{-(j_2+2)}(A_{11}^{-1})\frac{\Gamma(\eta_1^{(0)})(\Gamma(-\eta_1^{(\infty)})}{\Gamma(\varepsilon_1')\Gamma(-\varepsilon_4')} \\ K_{21} &= -q^{-(j_4+1)}(A_{21}^{-1})\frac{\Gamma(\eta_1^{(0)})\Gamma(\eta_1^{(\infty)})}{\Gamma(\varepsilon_0)\Gamma(2/(K+2))} \\ K_{12} &= -q^{(j_1+1)}(A_{12}^{-1})\frac{\Gamma(-\eta_1^{(0)})\Gamma(-\eta_1^{(\infty)})}{\Gamma(-\varepsilon_0)\Gamma(-2/(K+2))} \\ K_{22} &= -q^{0}(-A_{22}^{-1})\frac{\Gamma(-\eta_1^{(0)})\Gamma(\eta_1^{(\infty)})}{\Gamma(-\varepsilon_1')\Gamma(\varepsilon_4')}. \end{aligned} \tag{4.20}$$

For $(D2)^3$, $j_4=j_1+j_2$; the elements of the braid matrix for this case are

$$\begin{aligned}
K_{11} &= -q^{(j_1+1)}(A_{12}^{-1})\frac{\Gamma(\eta_3^{(0)})\Gamma(-\eta_1^{(\infty)})}{\Gamma(\varepsilon_1')\Gamma(-2/(K+2))}\\
K_{21} &= q^{0}(A_{22}^{-1})\frac{\Gamma(\eta_3^{(0)})\Gamma(\eta_1^{(\infty)})}{\Gamma(\varepsilon_0)\Gamma(\varepsilon_2')}\\
K_{12} &= q^{(j_2-1)}(A_{13}^{-1})\frac{\Gamma(-\eta_3^{(0)})\Gamma(-\eta_1^{(\infty)})}{\Gamma(-\varepsilon_0)\Gamma(-\varepsilon_2')}\\
K_{22} &= q^{j_4}(A_{23}^{-1})\frac{\Gamma(-\eta_3^{(0)})\Gamma(\eta_1^{(\infty)})}{\Gamma(-\varepsilon_1')\Gamma(2/(K+2))}.
\end{aligned} \tag{4.21}$$

In the following we want to discuss the case $(D1)$. Because dim $V_0=1$, it is not important what bases are chosen. But for compatibility with condition $(D2)$ we choose

$$u^{(0)}=u^{(1)}=u^{(\infty)} \qquad \text{for } (D1)^{1,3}$$
$$u^{(0)}=u^{(1)}=-u^{(\infty)} \qquad \text{for } (D1)^{2}$$

i.e.

$$S^{(\infty 0)}=\begin{cases} 1 & \text{for } (D1)^{1,3}\\ -1 & \text{for } (D1)^{2}\end{cases}. \tag{4.22}$$

The exponents of equation (2.6) at $\zeta=0,1,\infty$ for cases $(D1)$ are given as

$$\begin{aligned}
&(D1)^1 \qquad \gamma^{(0)}=\frac{2j_4+4}{k+2} \qquad \gamma^{(1)}=\frac{2j_2}{k+2} \qquad \gamma^{(20)}=\frac{-2j_1-2}{k+2}\\
&(D1)^2 \qquad \gamma^{(0)}=\frac{1}{k+2}(2j_4+4) \qquad \gamma^{(1)}=-\frac{1}{k+2}(2j_2+2) \qquad \gamma^{(\infty)}=\frac{2j_1}{k+2}\\
&(D1)^3 \qquad \gamma^{(0)}=\frac{1}{k+2}(2-2j_4) \qquad \gamma^{(1)}=\frac{2j_2}{k+2} \qquad \gamma^{(\infty)}=\frac{2j_1}{k+2}.
\end{aligned} \tag{4.23}$$

We write $\psi(\zeta)=u^{(i)}\psi^{(i)}(\zeta)$, where $\psi^{(0)}$, $\psi^{(1)}$ and $\psi^{(\infty)}$ satisfy the equations

$$\begin{aligned}
\frac{d}{d\zeta}\psi^{(i)} &= \left(\frac{\gamma^{(0)}}{\zeta}+\frac{\gamma^{(1)}}{\zeta-1}\right)\psi^{(i)} \qquad i=0,1\\
\frac{d}{d\eta}\psi^{(\infty)} &= \left(\frac{\gamma^{(\infty)}}{\eta}+\frac{\gamma^{(1)}}{\eta-1}\right)\psi^{(\infty)}
\end{aligned} \tag{4.24}$$

The solutions to (4.24) are

$$\begin{aligned}
\psi^{(0)} &= \psi^{(1)} = c\zeta^{\gamma^{(0)}}(1-\zeta)^{\gamma^{(1)}}\\
\psi^{(\infty)} &= c\eta^{\gamma^{(\infty)}}(1-\eta)^{\gamma^{(1)}}
\end{aligned} \tag{4.25}$$

For cases $(D1)^{1,3}$, we have $\psi^{(\infty)}=q^{-j_2}\psi^{(0)}$, and for $(D1)^2$ we have $\psi^{(\infty)}=q^{(j_2+1)}\psi^{(0)}$. From $S^{(0\infty)}\psi^{(0)}=\psi^{(\infty)}K$ we get the braid matrices for cases $(D1)^{1,2,3}$

$$K=\begin{cases} q^{j_2} & \text{for } (D1)^{1,3} \qquad (4.26a)\\ q^{-(j_2+1)} & \text{for } (D1)^{2}. \qquad (4.26b)\end{cases}$$

We have discussed the cases $(D2)$ and $(D1)$. We can see that the crossing matrix K depends only on the exponents of the equations at singular points. This is also valid for ($D3$). Since ($D2$) and ($D1$) are special cases of ($D3$) with some constraints, we can get the braid matrix for ($D3$) through that for ($D1$)[1,2,3] and ($D2$)[1,2,3].

First we introduce some quantities which are combinations of the exponents of $K-z$ equations for case ($D3$):

$$\begin{aligned}
&\alpha_1=\tfrac{1}{2}(\gamma_1^{(0)}+\gamma_1^{(1)}+\gamma_1^{(\infty)})=\varepsilon_0 && \alpha_2=-\tfrac{1}{2}(\gamma_3^{(0)}+\gamma_3^{(1)}+\gamma_3^{(\infty)})=\varepsilon_0'\\
&\beta_1^{(0)}=\tfrac{1}{2}(\gamma_1^{(0)}+\gamma_1^{(1)}+\gamma_3^{(\infty)})=\varepsilon_1 && \beta_2^{(0)}=-\tfrac{1}{2}(\gamma_3^{(0)}+\gamma_3^{(1)}+\gamma_1^{(\infty)})=\varepsilon_1'\\
&\gamma_1=\tfrac{1}{2}(\gamma_1^{(0)}+\gamma_1^{(\infty)}+\gamma_3^{(1)})=\varepsilon_2 && \gamma_2=-\tfrac{1}{2}(\gamma_3^{(0)}+\gamma_3^{(\infty)}+\gamma_1^{(1)})=\varepsilon_2'\\
&\beta_1^{(\infty)}=\tfrac{1}{2}(\gamma_1^{(\infty)}+\gamma_1^{(1)}+\gamma_3^{(0)})=\varepsilon_4 && \beta_2^{(\infty)}=-\tfrac{1}{2}(\gamma_3^{(\infty)}+\gamma_3^{(1)}+\gamma_1^{(0)})=\varepsilon_4'.
\end{aligned} \tag{4.27}$$

Then the elements of K for ($D3$) are given by

$$\begin{aligned}
K_{11}&=q^{-(j_4+j_1+2)}\left(\frac{\eta_1^{(0)}\eta_2^{(0)}\eta_1^{(\infty)}\eta_2^{(\infty)}}{\beta_1^{(0)}\beta_2^{(0)}\beta_1^{(\infty)}\beta_2^{(\infty)}}\right)^{1/2}\frac{\Gamma(\eta_1^{(0)})\Gamma(\eta_2^{(0)})\Gamma(-\eta_1^{(\infty)})\Gamma(-\eta_2^{(\infty)})}{\Gamma(\beta_1^{(0)})\Gamma(\beta_2^{(0)})\Gamma(-\beta_1^{(\infty)})\Gamma(-\beta_2^{(\infty)})}\\
K_{21}&=-q^{-(j_4+1)}\left(\frac{\eta_1^{(0)}\eta_2^{(0)}\eta_1^{(\infty)}\eta_3^{(\infty)}}{2\alpha_1\beta_1^{(0)}\gamma_1\beta_2^{(\infty)}}\right)^{1/2}\frac{\Gamma(\eta_1^{(0)})\Gamma(\eta_2^{(0)})\Gamma(\eta_1^{(\infty)})\Gamma(-\eta_3^{(\infty)})\Gamma(1/(k+2))}{\Gamma(\alpha_1)\Gamma(\beta_1^{(0)})\Gamma(\gamma_1)\Gamma(-\beta_2^{(\infty)})\Gamma(2/(K+2))}\\
K_{31}&=q^{(j_1-j_4-1)}\left(\frac{\eta_1^{(0)}\eta_2^{(0)}\eta_2^{(\infty)}\eta_3^{(\infty)}}{\alpha_1\alpha_2\gamma_1\gamma_2}\right)^{1/2}\frac{\Gamma(\eta_1^{(0)})\Gamma(\eta_2^{(0)})\Gamma(\eta_2^{(\infty)})\Gamma(\eta_3^{(\infty)})}{\Gamma(\alpha_1)\Gamma(\alpha_2)\Gamma(\gamma_1)\Gamma(\gamma_2)}\\
K_{12}&=-q^{(j_1+1)}\left(\frac{\eta_1^{(0)}\eta_3^{(0)}\eta_1^{(\infty)}\eta_2^{(\infty)}}{2\alpha_1\beta_2^{(0)}\gamma_1\beta_1^{(\infty)}}\right)^{1/2}\frac{\Gamma(-\eta_1^{(0)})\Gamma(\eta_3^{(0)})\Gamma(-\eta_1^{(\infty)})\Gamma(-\eta_2^{(\infty)})\Gamma(-1/(K+2))}{\Gamma(-\alpha_1)\Gamma(\beta_2^{(0)})\Gamma(-\gamma_1)\Gamma(-\beta_2^{(\infty)})\Gamma(-2/(K+2))}\\
K_{22}&=\left(\frac{\sqrt{\eta_1^{(0)}\eta_3^{(0)}\eta_1^{(\infty)}\eta_3^{(\infty)}}}{\alpha_1\gamma_2\Gamma(\alpha_1)\Gamma(-\alpha_1)(\Gamma(\gamma_2)\Gamma(-\gamma_2)}-\frac{\sqrt{\eta_1^{(0)}\eta_3^{(0)}\eta_1^{(\infty)}\eta_3^{(\infty)}}}{\beta_1^{(0)}\beta_2^{(\infty)}\Gamma(\beta_1^{(0)})\Gamma(-\beta_1^{(0)})\Gamma(\beta_2^{(\infty)})\Gamma(-\beta_2^{(\infty)})}\right)\\
&\qquad\times\Gamma(-\eta_1^{(0)})\Gamma(\eta_3^{(0)})\Gamma(\eta_1^{(\infty)})\Gamma(-\eta_3^{(\infty)}) \qquad\qquad (4.28)\\
K_{32}&=-q^{j_1}\left(\frac{\eta_1^{(0)}\eta_3^{(0)}\eta_2^{(\infty)}\eta_3^{(\infty)}}{2\alpha_2\beta_1^{(0)}\gamma_2\beta_2^{(\infty)}}\right)^{1/2}\frac{\Gamma(-\eta_1^{(0)})\Gamma(\eta_3^{(0)})\Gamma(\eta_2^{(\infty)})\Gamma(\eta_3^{(\infty)})\Gamma(-1/(K+2))}{\Gamma(\alpha_2)\Gamma(-\beta_1^{(0)})\Gamma(\gamma_2)\Gamma(\beta_2^{(\infty)})\Gamma(-2/(k+2))}\\
K_{13}&=q^{(j_4-j_1-1)}\left(\frac{\eta_2^{(0)}\eta_3^{(0)}\eta_1^{(\infty)}\eta_2^{(\infty)}}{\alpha_1\alpha_2\gamma_1\gamma_2}\right)^{1/2}\frac{\Gamma(-\eta_2^{(0)})\Gamma(-\eta_3^{(0)})\Gamma(-\eta_1^{(\infty)})\Gamma(-\eta_2^{(\infty)})}{\Gamma(-\alpha_1)\Gamma(-\alpha_2)\Gamma(-\gamma_1)\Gamma(-\gamma_2)}\\
K_{23}&=-q^{j_4}\left(\frac{\eta_2^{(0)}\eta_3^{(0)}\eta_1^{(\infty)}\eta_3^{(\infty)}}{2\alpha_2\beta_2^{(0)}\gamma_2\beta_1^{(\infty)}}\right)^{1/2}\frac{\Gamma(-\eta_2^{(0)})\Gamma(-\eta_3^{(0)})\Gamma(\eta_1^{(\infty)})\Gamma(-\eta_3^{(\infty)})\Gamma(1/(K+2))}{\Gamma(-\alpha_2)\Gamma(-\beta_2^{(0)})\Gamma(\beta_1^{(\infty)})\Gamma(-\gamma_2)\Gamma(2/(K+2))}\\
K_{33}&=q^{j_1+j_4}\left(\frac{\eta_2^{(0)}\eta_3^{(0)}\eta_2^{(\infty)}\eta_3^{(\infty)}}{\beta_1^{(0)}\beta_2^{(0)}\beta_1^{(\infty)}\beta_2^{(\infty)}}\right)^{1/2}\frac{\Gamma(-\eta_2^{(0)})\Gamma(-\eta_3^{(0)})\Gamma(\eta_2^{(\infty)})\Gamma(\eta_3^{(\infty)})}{\Gamma(-\beta_1^{(0)})\Gamma(-\beta_2^{(0)})\Gamma(\beta_1^{(\infty)})\Gamma(\beta_2^{(\infty)})}.
\end{aligned}$$

How can we know the expression for K for case ($D3$) is given by (4.28)? We have the following explanation. For example, the K_{11} elements of matrix K for case ($D3$) should degenerate to K_{11} of equation (4.20) when $j_2=j_1+j_4$, and degenerate to (4.26b) when $j_2=j_1+j_4+1$, which means that the K_{11} for ($D3$) has the form given in (4.28); the K_{21} of K for ($D3$) is determined by its degenerate form K_{11} in (4.19) and K_{21} in (4.20). Other elements of K for ($D3$) can also be determined by such a method.

In the foregoing, we have discussed the braid matrix K for $j_3=1$. A similar method can also be used to determine the crossing matrices of ($D4$) for $j_3=\frac{3}{2}$ from their degenerate forms in ($D3$)[1,2,3], ($D2$)[1,2,3] and ($D1$)[1,2,3]. For generic j_3, the crossing matrices for ($D(2j_3+1)$) can be determined from their form in the CG-degenerate cases $(D(2j_3))^{1,2,3}$ $(D(2j_3-1))^{1,2,3},\ldots,(D1)^{1,2,3}$. The braid matrix $K(j_4j_3j_2j_1)$ for $(D(2j_3+1))$ has the form

$$K(j_4j_3j_2j_1)=(K_{ll'}) \qquad l,l'=0,1,\ldots,2j_3. \tag{4.29}$$

The elements are

$$\begin{aligned}K_{ll'}=(-1)^{j_2-j_4+2j_3-l'-l}&q^{(-j_3(j_1+j_4)+l'j_4+lj_1-j_3^2+j_3(l+l'-1)-(1/2)l(l-1)-(1/2)l'(l'-1))}\\&\times[(2j_4+2j_3-2l'+1)(2j_1+2j_3-2l+1)]^{-1/2}\\&\times[\Gamma((2j_4+2j_3-2l'+1)/(K+2))\Gamma(-(2j_1+2j_3-2l+1)/(K+2)]^{-1}\\&\times\begin{Bmatrix}j_2 & j_1 & j_4+j_3-l'\\ j_3 & j_4 & j_1+j_3-l\end{Bmatrix}_\Gamma.\end{aligned} \tag{4.30}$$

The Γ $6j$ symbol on the right-hand side of (4.30) has the following definition:

$$\begin{aligned}&\begin{Bmatrix}j_2 & j_1 & j_4+j_3-l'\\ j_3 & j_4 & j_1+j_3-l\end{Bmatrix}\\&\quad=\left(\frac{P_\oplus(j_1+j_2-j_3-j_4+l')P_\oplus(j_1-j_2+j_3+j_4-l')P_\oplus(-j_1+j_2+j_3+j_4-l')}{P_\ominus(j_1+j_2+j_3+j_4-l'+1)}\right.\\&\qquad\times\frac{P_\ominus(-j_1+j_2-j_3+j_4+l)P_\ominus(j_1-j_2+j_4+j_3-l)P_\ominus(j_1+j_2-j_4+j_3-l)}{P_\oplus(j_1+j_2+j_3+j_4-l+1)}\\&\qquad\left.\times\frac{P_\ominus(l)P_\ominus(2j_3-l)P_\ominus(2j_1-l)}{P_{\oplus l}(2j_1+2j_3-l+1)}\,\frac{P_\oplus(l')P_\oplus(2j_3-l')P_\oplus(2j_4-l')}{P_{\ominus l'}(2j_3+2j_4-l'+1)}\right)^{1/2}\\&\qquad\times\sum_{z\geqslant0}\frac{(-1)^zP_{\oplus\ominus}(z+1)}{P_{\oplus\ominus}(z-j_1-j_2-j_3-j_4+l')P_{\oplus\ominus}(z-j_1-j_2-j_3-j_4+l)}\\&\qquad\times\frac{1}{P_{\oplus\ominus}(z-2j_1-2j_3+l)P_{\oplus\ominus}(z-2j_4-2j_3+l')P_{\oplus\ominus}(j_1+j_2+j_3+j_4-z)}\\&\qquad\times\frac{1}{P_{\oplus\ominus}(2j_1+2j_3+2j_4-l'-l-z)P_{\oplus\ominus}(j_1+j_2+j_4+3j_3-l-l'-z)}\end{aligned} \tag{4.31}$$

where

$$\begin{aligned}&P_\oplus(j)=\left(\prod_{n=1}^{j}(n\Gamma^2(n/(k+2))\right)^{-1} \qquad P_\ominus=\left(\prod_{n=1}^{j}(-n\Gamma^2(n/(k+2))\right)^{-1}\\&P_{\oplus\ominus}(j)=\left(\prod_{n=1}^{j}(n\Gamma(-n/(k+2))\Gamma(n/(k+2)))\right)^{-1}\\&P_{\oplus l}(j)=\left(\prod_{1}^{j-l}(n\Gamma^2(-n/(k+2)))\prod_{j-l}^{j}(n\Gamma^2(n/(k+2)))\right)^{-1}\\&P_{\ominus l}(j)=\left(\prod_{1}^{j-l}(n\Gamma^2(n/(k+2)))\prod_{j-l}^{j}(n\Gamma^2(-n/(k+2)))\right)^{-1}\end{aligned} \tag{4.32}$$

and all $P(0)=1$.

Now let us discuss the expressions for K under the KCG constraints. The expression for $K(j_4j_3j_2j_1)$ given by (4.30) is a KCG free case. But the matrices of KCG constraint cases can be given by some submatrices of K given by (4.30).

The K matrices of CG constraint cases (DM) are given by the submatrices as follows

$$K[(DM)]=(K_{ll'}) \tag{4.33}$$

for $(DM)^1$

$$\begin{cases} l = 2j_3 - m + 1, 2j_3 - m + 2, \ldots, 2j_3 \\ l' = 0, 1, \ldots m - 1 \end{cases} \tag{4.34}$$

for $(DM)^2$

$$\begin{cases} l = 0, 1, \ldots, m - 1 \\ l' = 0, 1, \ldots, m - 1, \end{cases} \tag{4.35}$$

and for $(DM)^3$

$$\begin{cases} l = 0, 1, \ldots, m - 1 \\ l' = 2j_3 - m + 1, 2j_3 - m + 2, \ldots, 2j_3. \end{cases} \tag{4.36}$$

When the level k takes certain values in the region $j_1+j_2-j_3+j_4<k<j_1+j_2+j_3+j_4$ the k constraint condition must be taken into consideration. In the k constraint case, because of trunction of the intermediate edges, some of the elements defined in (4.30) are not admissible, the admissible elements provide the braid matrix for this case

$$K(k \text{ constraint case}) = (K_{ll'}) \tag{4.37}$$

$$l, l' = j_1+j_2+j_3+j_4-k, j_1+j_2+j_3+j_4-k+1, \ldots, 2j_3. \tag{4.38}$$

In this case, the non-admissible elements in (4.30) may develop poles, but the behaviour of the admissible elements is good.

So far in this section we have discussed the braid matrices K of the model. Similar procedure can also be done for determining the fusion matrices F, which have the form (for the KCG free case)

$$F(j_4 j_3 j_2 j_1) = (F_{ll'}) \qquad l, l' = 0, 1, \ldots, 2j_3 \tag{4.39}$$

$$F_{ll'} = [(2j_4+2j_3-2l'+1)(2j_2+2j_3-2l+1)]^{-1/2}(\Gamma((2j_4+2j_3-2l'+1)/(k+2)) \times \Gamma(-(2j_2+2j_3-2l+1)/(k+2)))^{-1} \begin{Bmatrix} j_1, & j_2 & j_4+j_3-l' \\ j_3 & j_4 & j_2+j_3-l \end{Bmatrix}_\Gamma . \tag{4.40}$$

From (4.30) and (4.40), we obtain the relation between K and F

$$K_{p\bar{p}}(j_4 j_3 j_2 j_1) = (-1)^{p+\bar{p}-j_1-j_4} q^{(c_{j_1}+c_{j_4}-c_p-c_{\bar{p}})/2} F_{p\bar{p}}(j_4 j_3 j_1 j_2) \tag{4.41}$$

where we use the intermediate edges p and $\bar{p}$ to present indices of the matrices, and $c_j = j(j+1)$.

5. Relation between the SU(2) wzw model and the SL(2, q) quantum group

The braid matrix of the SL(2, q) quantum group has the form [5]

$$c_{p\bar{p}}\begin{bmatrix} j_3 & j_2 \\ j_4 & j_1 \end{bmatrix} = (-1)^{p+\bar{p}-j_1-j_4} q^{[c_{j_1}+c_{j_4}-c_p-c_{\bar{p}}]} \begin{pmatrix} j_3 & j_4 & p \\ j_2 & j_1 & \bar{p} \end{pmatrix}_q \tag{5.1}$$

the last factor on the right-hand side is the quantum $6j$ coefficient [8]. In [5] it is pointed out that the braid matrix $K(j\frac{1}{2}\frac{1}{2}j)$ of the $SU(2)_K$ wzw model and the braid matrix $C[{}^{1/2}_{j}\,{}^{1/2}_{j}]$ of SL(2, q) are connected by a similarity transformation. But we find that in general cases, $K(j_4 j_3 j_2 j_1)$ and $C[{}^{j_3 j_2}_{j_4 j_1}]$ are not related by similarity transformations, but by triangle matrix transformations. From the expression for $K(j_4\frac{1}{2}j_2 j_1)$ given in [7], we obtain

$$k(j_4\tfrac{1}{2}j_2j_1)=(-1)^{2(j_1+j_2-j_4+1/2)}\begin{pmatrix}\gamma_+^{-1} & \\ & \gamma_-^{-1}\end{pmatrix}C^{\mathrm{T}}\begin{bmatrix}\frac{1}{2} & j_2\\ j_4 & j_1\end{bmatrix}\begin{pmatrix}\beta_+ & \\ & \beta_-\end{pmatrix} \tag{5.2}$$

where

$$\gamma_\pm=\Gamma\left(\pm\frac{2j_1+1}{k+2}\right)\left(\frac{\Gamma(\pm(-j_1+j_2+j_4+\frac{1}{2})/(k+2)}{\Gamma(\pm(j_1+j_2+j_4+\frac{3}{2})/(k+2))\Gamma(\pm(j_1-j_2+j_4+\frac{1}{2})/(k+2))\,\Gamma(\pm(j_1+j_2-j_4+\frac{1}{2})/(k+2))}\right)^{1/2}$$

$$\beta_+=\left(\frac{\Gamma((2j_4+1)/(k+2))\Gamma((2j_1+1)/(k+2))\,\Gamma(-(2j_1+1)/(k+2))\Gamma(-(-j_1+j_2+j_4+\frac{1}{2})/(k+2))}{\Gamma(-(2j_4+1)/(k+2))\Gamma((j_1+j_2+j_4+\frac{3}{2}/(k+2))\,\Gamma((j_1+j_2+j_4+\frac{1}{2})/(k+2))\Gamma(-(j_1+j_2-j_4+\frac{1}{2})/(k+2))}\right)^{1/2} \tag{5.3}$$

$$\beta_-=\left(\frac{\Gamma(-(2j_4+1)/(k+2))\Gamma((2j_1+1)/(k+2))\,\Gamma(-(2j_1+1)/(k+2))\Gamma((-j_1+j_2+j_4+\frac{1}{2})/(k+2))}{\Gamma((2j_4+1)/(k+2))\Gamma((j_1+j_2+j_4+\frac{3}{2})/(k+2))\,\Gamma(-(j_1-j_2+j_4+\frac{1}{2})/(k+2))\Gamma((j_1+j_2+j_4+\frac{1}{2})/(k+2))}\right)^{1/2}.$$

When $j_1=j_4$, $\gamma_\pm=\beta_\pm$, this triangle matrix transformation becomes a similarity transformation. For $j_3=1$, we have

$$k(j_41j_2j_1)=(-1)^{2(j_1+j_2-j_4+1)}\begin{pmatrix}\alpha_+^{-1} & & \\ & \alpha_0^{-1} & \\ & & \alpha_-^{-1}\end{pmatrix}C^{\mathrm{T}}\begin{bmatrix}1 & j_2\\ j_4 & j_1\end{bmatrix}\begin{pmatrix}\beta_+ & & \\ & \beta_0 & \\ & & \beta_-\end{pmatrix} \tag{5.4}$$

where

$$\alpha_\pm=\Gamma(\pm\eta_2^{(\infty)})\left(\frac{\Gamma(\pm\eta_1^{(\infty)})\Gamma(\pm\eta_3^{(\infty)})\Gamma(\pm\beta_1^{(0)})\Gamma(\pm\beta_2^{(0)})}{\Gamma(\pm\alpha_1)\Gamma(\pm\alpha_2)\Gamma(\pm\gamma_1)\Gamma(\pm\gamma_2)\Gamma(\pm\beta_1^{(\infty)})\beta(\pm\beta_2^{(\infty)})}\right)^{1/2}$$

$$\alpha_0=\frac{\Gamma(-\eta_1^{(\infty)})\Gamma(\eta_3^{(\infty)})}{\Gamma(\eta_1^{(\infty)})\Gamma(-\eta_3^{(\infty)})}$$

$$\times\left(\frac{\Gamma(\eta_1^{(\infty)})\Gamma(-\eta_2^{(\infty)})\Gamma(\eta_2^{(\infty)})\Gamma(-\eta_3^{(\infty)})\Gamma(\beta_1^{(0)})\Gamma(-\beta_2^{(0)})\Gamma(-1/(k+2))\Gamma(2/(k+2))}{\Gamma(-\alpha_1)\Gamma(\alpha_2)\Gamma(-\gamma_1)\Gamma(\gamma_2)\Gamma(-\beta_1^{(\infty)})\Gamma(\beta_2^{(\infty)})\Gamma(1/(k+2))\Gamma(-2/(k+2)}\right)^{1/2}$$

$$\beta_+=\left(\frac{\Gamma(-\eta_1^{(\infty)})\Gamma(-\eta_2^{(\infty)})\Gamma(\eta_2^{(\infty)})\Gamma(\eta_3^{(\infty)})\Gamma(\eta_1^{(0)})\Gamma(-\eta_2^{(0)})\Gamma(-\beta_1^{(0)})\Gamma(-\beta_2^{(0)})}{\Gamma(-\eta_1^{(0)})\Gamma(-\eta_2^{(0)})\Gamma(\alpha_1)\Gamma(\alpha_2)\Gamma(\gamma_1)\Gamma(\gamma_2)\Gamma(-\beta_1^{(\infty)})\Gamma(-\beta_2^{(\infty)})}\right)^{1/2} \tag{5.5}$$

$$\beta_0=\left(\frac{\Gamma(-\eta_1^{(0)})\Gamma(\eta_3^{(0)})\Gamma(-\eta_1^{(\infty)})\Gamma(-\eta_2^{(\infty)})\Gamma(\eta_2^{(\infty)})\,\Gamma(\eta_3^{(\infty)})\Gamma(\beta_1^{(0)})\Gamma(-\beta_2^{(0)})\Gamma(-1/(k+2))\Gamma(2/(k+2)}{\Gamma(\eta_1^{(0)})\Gamma(-\eta_3^{(0)})\Gamma(-\alpha_1)\Gamma(\alpha_2)\Gamma(-\gamma_1)\Gamma(\gamma_2)\,\Gamma(-\beta_1^{(\infty)})\Gamma(\beta_2^{(\infty)})\Gamma(1/(k+2))\Gamma(-2/(k+2))}\right)^{1/2}$$

$$\beta_-=\left(\frac{\Gamma(-\eta_2^{(0)})\Gamma(-\eta_3^{(0)})\Gamma(-\eta_1^{(\infty)})\Gamma(-\eta_2^{(\infty)})\Gamma(\eta_2^{(\infty)})\Gamma(\eta_3^{(\infty)})\Gamma(\beta_1^{(0)})\Gamma(\beta_2^{(0)})}{\Gamma(\eta_2^{(0)})\Gamma(\eta_3^{(0)})\Gamma(-\alpha_1)\Gamma(-\alpha_2)\Gamma(-\gamma_1)\Gamma(-\gamma_2)\Gamma(\beta_1^{(\infty)})\Gamma(\beta_2^{(\infty)})}\right)^{1/2}.$$

When $j_1=j_4$, $\alpha_\pm=\beta_\pm$, $\alpha_0=\beta_0$, (5.4) becomes a similarity transformation. For generic j_3, we also find similar relations. This is mainly because the elements of the matrix K

given by (4.30) and the quantum $6j$ coefficients are related as follows:

$$K_{ll'} = (-1)^{j_2-j_4+2j_3-l'-l} q^{-j_3(j_1+j_4)+l'j_4+lj_1-j_3^2+j_3(l+l'-1)-(1/2)l(l-1)-(1/2)l'(l'-1)}$$

$$\times\sqrt{[2j_4+2j_3-2l'+1][2j_1+2j_3-2l+1]}\begin{Bmatrix} j_2 & j_1 & j_4+j_3-l' \\ j_3 & j_4 & j_1+j_3-l \end{Bmatrix}_q f(\Gamma) \tag{5.6}$$

where

$$[n] = \frac{q^{n/2}-q^{-n/2}}{q^{1/2}-q^{-1/2}}. \tag{5.7}$$

The function $f(\Gamma)$ is square root of some products of ratios of Γ functions:

$$f(\Gamma) = \left(\frac{\Gamma(-(2j_4+2j_3-2l'+1)/(k+2))\Gamma((2j_1+2j_3-2l+1)/(k+2))}{\Gamma((2j_4+2j_3-2l'+1)/(k+2))\Gamma(-(2j_1+2j_3-2l+1)/(k+2))}\right.$$

$$\times\frac{\rho(j_1+j_2+j_3+j_4-l+1)}{\rho(j_1+j_2+j_3+j_4-l'+1)}\frac{\rho(j_2+j_4-j_1-j_3+l)}{\rho(-j_1+j_2+j_3+j_4-l')}$$

$$\times\frac{\rho(j_1-j_2+j_3+j_4-l)}{\rho(j_1-j_2+j_3+j_4-l')}\frac{\rho(j_1+j_2+j_3-j_4-l)}{\rho(j_1+j_2-j_3-j_4+l')}$$

$$\left.\times\frac{\rho(l)}{\rho(l')}\frac{\rho(2j_3-l)}{\rho(2j_3-l')}\frac{\rho(2j_1-l)}{\rho(2j_4-l')}\frac{\rho^2(2j_4+2j_3-2l'+1)}{\rho^2(2j_1+2j_3-2l+1)}\frac{\rho(2j_1+2j_3-l+1)}{\rho(2j_4+2j_3-l'+1)}\right)^{1/2} \tag{5.8}$$

where

$$\rho^{(j)} = \prod_{n=1}^{j}\frac{\Gamma(n/(k+2))}{\Gamma(-n/(k+2))} \qquad \rho(0)=1. \tag{5.9}$$

From (5.6) we can see that the braid matrix of the SU(2)$_k$ wzw model and that of the SL(2, q) quantum group differ by a factor $f(\Gamma)$. From the discussion in [10], we know that $f(\Gamma)$ is dependent on the structure constant of the SU(2)$_k$ wzw model. If the conformal block $\psi(\zeta)$ is not normalised to 1, but normalised to the structure constant, then the crossing matrices of the SU(2) wzw model become identical to those of the SL(2, q) quantum group.

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